Distributed Algorithms

Logic 101 - solutions
1st exercise session

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Example 1 (Conditional statements)

Write the converse, contrapositive and inverse of the following sentence:

“If process x fails, then process y never receives message m”

Reminder:

Let P be the proposition $p \rightarrow q$:

- The *converse* of P is: $q \rightarrow p$:
- The inverse of P is: $\neg p \rightarrow \neg q$
- The contrapositive of P is: $\neg q \rightarrow \neg p$

Notes:

- Only the contrapositive of a conditional statement is equivalent to it.
- The proposition “p iff q” means that both P and the converse of P are true.

Exercise 1 (Conditional statements)

Write the negation of the following sentence:

"If process x fails, then process y never receives message m"

Reminder:

Let P be a proposition. The negation of P is ¬P ("not P"). For example:

- ¬"7 is odd" = "7 is not odd" = "7 is even" (if you prove it!)
- ¬"All cats are animals" = "Some cats are not animals"

Hints:

- The negation of ¬p → ¬q is *not* p → ¬q.
- Express the implication in terms of *and* and *or* expressions.

Example 2

If the following statement is true:

If process i fails, then instantly all processes j≠i fail

Which of the following are also true?

1. If a process j≠i fails, then process i has failed,
2. If a process j≠i fails, nothing can be said about process i,
3. If a process j≠i fails, then process i has not failed

(continues on next slide)

Example 2 (contd)

4. If no process $j \neq i$ fails, nothing can be said about process i,
5. If no process $j \neq i$ fails, then process i has failed,
6. If no process $j \neq i$ fails, then process i has not failed,
7. If all processes $j \neq i$ fail, then process i has failed,
8. If all processes $j \neq i$ fail, nothing can be said about process i,
9. If all processes $j \neq i$ fail, then process i has not failed,
10. If some process $j \neq i$ does not fail, nothing can be said about process i,
11. If some process $j \neq i$ does not fail, then process i has failed,
12. If some process $j \neq i$ does not fail, then process i has not failed.

Exercise 2

Replace “instantly” with “eventually” in Example 2.

Exercise 2 (solution)

1. **False:** Some process j can fail for a reason not related to the failure of process i.
2. **True:** explanation in (1).
3. **False:** explanation in (1).
4. **True:** Because of “eventually”.
5. **False.**
6. **False:** Because of “eventually”.
7. **False.**
8. **True:** Nothing can be said about process i.
9. **False.**
10. **True:** Nothing can be said about process i, because of “eventually”.
11. **False.**
12. **False:** Nothing can be said about process i, because of “eventually”.

Example 3 (Proof by cases)

Let x, y, z, q be natural numbers such that

$$x^2 + 5y^2 + 5z^2 = q^2$$

Prove that q is even if and only if all of $x, y,$ and z are even as well.

Exercise 3 (Proof by cases)

Prove that $x + |x - 7| \geq 7$

Exercise 3 (solution)

For the set of real numbers, we know that:

- $|a| = -a$, if $a < 0$
- $|a| = a$, if $a \geq 0$

So:

- If $x < 7$: $|x - 7| = 7 - x$, therefore $x + |x - 7| = x + (7 - x) = 7 \geq 7$
- If $x \geq 7$: $|x - 7| = x - 7$, therefore $x + |x - 7| = x + (x - 7) = 2x - 7 \geq 2*7 - 7 \rightarrow x + |x - 7| \geq 7$

Example 4 (Proof by contradiction)

Prove that the set of prime numbers is infinite.

Exercise 4 (Proof by contradiction)

Prove that if α^2 is even, α is even.

Exercise 4 (solution)

When we want to prove something by contradiction, we start by assuming that the negation (of whatever we are trying to prove) is true.

We said in the classroom that $p \rightarrow q$ is equivalent to $\neg p \lor q$.
Therefore the negation of $p \rightarrow q$ is $p \land \neg q$.

With that said, let's assume that a^2 is even and a is odd. Since a is odd, a can be written as $a=2k+1$.
Therefore, $a^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Thus, a^2 is odd, a contradiction!

Bonus Exercise 4 (Proof by contradiction)

Prove that $\sqrt{2}$ is irrational.

Hint: Use the result of exercise 4

Proof by contradiction:

- In order to prove p, find a contradiction q such that $\neg p \rightarrow q$ is true.
- A contradiction always has the form: $q \equiv r \wedge \neg r$.

Hints:

- Use the result of Exercise 3.
- A rational number is always in the form r/q, where r is integer, q is natural, and r and q *have no common divisor*.

Exercise 4 (Solution)

- Assume that $\sqrt{2}$ is rational, i.e. $\sqrt{2} = a/b$ where a,b are coprime (have no common divisors).
- We square both sides, thus $2 = a^2/b^2 \rightarrow a^2 = 2b^2$.
- Therefore, a^2 is even, and using the result of the previous exercise we know that a is even.
- Since a is even, it has the form $a=2k$. We substitute this in the previous equation and we have that:
- $(2k)^2 = 2b^2 \rightarrow 4k^2 = 2b^2 \rightarrow b^2 = 2k^2$.
- Since $b^2 = 2k^2$, this means that b^2 is even $\rightarrow$ b is even, which is a contradiction!
- The contradiction is that we assumed a,b to be coprime, but we concluded that both are even!

Example 5 (proof by induction)

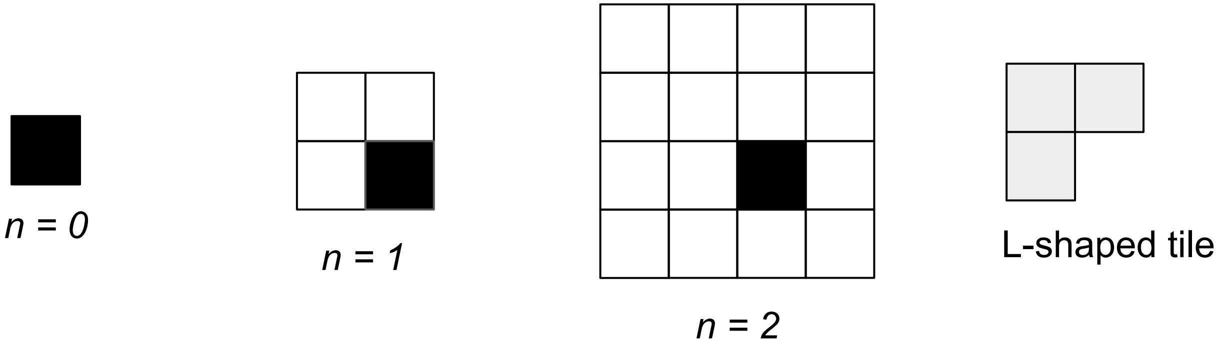
Prove that $11^n - 6$ is divisible by 5, for every positive integer n.

Exercise 5 (proof by induction)

A chessboard of size 2^nx2^n ($n \geq 0$) has all of its squares painted white, except for one arbitrary square, which is painted black.

Prove that for every $n \geq 0$, you can cover all the white squares of the chessboard with L-shaped non-overlapping tiles.

e.g.

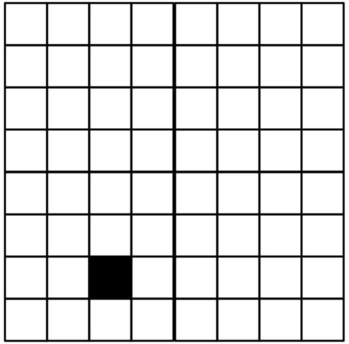


Exercise 5 (solution 1/2)

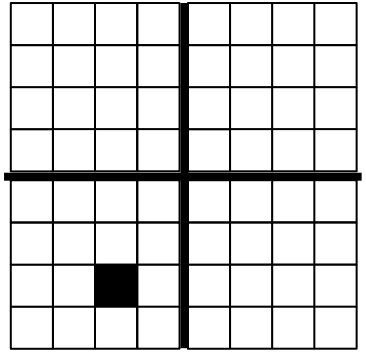
We will use induction:

- Base case ($n=0$): We can tile one black square, using 0 L-shaped tiles.
- Inductive step: Suppose this property holds for $n \geq 0$:
  - i.e., we can tile a $2^n x 2^n$ grid using L-shaped tiles, leaving a single square uncovered (the black square) at an *arbitrary* location. We will show how to tile a $2^{n+1} x 2^{n+1}$ grid.

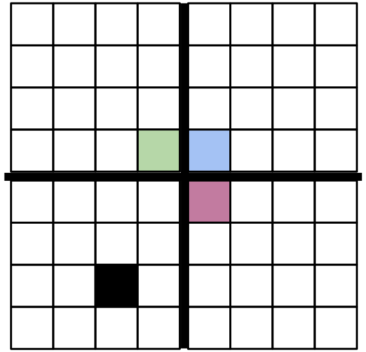
Exercise 5 (solution 2/2)



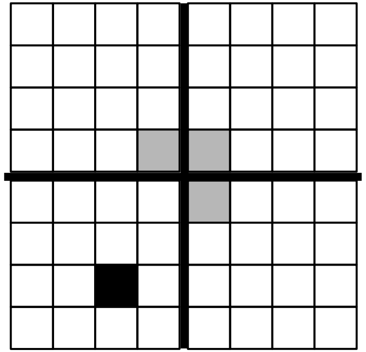
Suppose the grid has size $2^{n+1}x2^{n+1}$ (we show a grid for n=3) and the black square is somewhere in the grid.



We split the $2^{n+1}x2^{n+1}$ grid in 4 sub-grids of size 2^nx2^n.



We can tile each sub-grid because of the inductive step. For the top-left sub-grid we leave the green square uncovered. We also leave the blue and the red squares uncovered in their corresponding sub-grids.



For the three squares in the middle, we can use an L-shaped tile.

Bonus Exercise 5 (proof by induction)

Consider a country with $n \geq 2$ cities. For every pair of different cities x, y, there exists a direct route (single direction) either from x to y or from y to x. Show that there exists a city that we can reach from every other city either directly or through exactly one intermediate city.

Exercise 5 (solution 1/2)

We name “central” the city that we can reach from every other city either directly or through exactly one intermediate city.

Base case ($n=2$): It obviously holds. Either one of the cities is “central”.

Inductive step: Suppose this property holds for $n \geq 2$ cities. We will prove that it will still hold for $n+1$ cities.

Exercise 5 (solution 2/2)

Let n+1 cities, c_i, i=0, …, n, where for every pair of different cities c_i, c_j, there exists a direct route (single direction) either from c_i to c_j or from c_j to c_i.

We consider only the first n cities, i.e. cities c_i, i=0, …, n-1. According to the inductive step, there exists one central city among these n cities. Let c_j be that city.

We now exclude city c_j and consider the rest of the cities. Again, we have n cities, therefore there should exist one city among them that is central. Let c_k be that city.

All cities apart from c_j and c_k can reach c_j and c_k either directly or through one intermediate city.

Furthermore, there exists a route between c_j and c_k:

- If the route is directed from c_j to c_k, then c_k is the central city for the n+1 cities.
- If the route is directed from c_k to c_j, then c_j is the central city for the n+1 cities.